

ROOTED TREES, NON-ROOTED TREES AND HAMILTONIAN B-SERIES

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ABSTRACT. We explore the relationship between (non-planar) rooted trees and free trees, i.e. without root. We give in particular, for non-rooted trees, a substitute for the Lie bracket given by the antisymmetrization of the pre-Lie product.

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1. INTRODUCTION

A striking link between rooted trees and vector fields on an affine space \mathbb{R}^n has been established by A. Cayley [8] as early as 1857. The interest for this correspondence has been renewed since J. Butcher showed the key role of rooted trees for understanding Runge-Kutta methods in numerical approximation [5, 4, 16]. The modern approach to this correspondence can be summarized as follows: the product on vector fields on \mathbb{R}^n defined by:

$$(1) \quad \left(\sum_{i=1}^n f_i \partial_i \right) \triangleright \left(\sum_{j=1}^n g_j \partial_j \right) := \sum_{j=1}^n \left(\sum_{i=1}^n f_i (\partial_i g_j) \right) \partial_j$$

is left pre-Lie, which means that for any vector fields a, b, c the associator $a \triangleright (b \triangleright c) - (a \triangleright b) \triangleright c$ is symmetric with respect to a and b . On the other hand, the free pre-Lie algebra with one generator (on some base field k) is the vector space \mathcal{T} spanned by the planar rooted trees [10, 15]. The generator is the one-vertex tree \bullet , and the pre-Lie product on rooted trees is given by grafting:

$$(2) \quad s \rightarrow t = \sum_{v \in \mathcal{V}(t)} s \rightarrow_v t,$$

where $s \rightarrow_v t$ is the rooted tree obtained by grafting the rooted tree s on the vertex v of the tree t . Hence for any vector field a on \mathbb{R}^n there exists a unique pre-Lie algebra morphism \mathcal{F}_a from \mathcal{T} to vector fields such that $\mathcal{F}_a(\bullet) = a$. This can be generalized to an arbitrary number of generators, since the free pre-Lie algebra on a set D of generators is the span of rooted trees with vertices coloured by D . In this case, for any collection $\underline{a} = (a_d)_{d \in D}$ of vector fields, there exists a unique pre-Lie algebra morphism $\mathcal{F}_{\underline{a}}$ from the linear span \mathcal{T}_D of coloured trees to vector fields on \mathbb{R}^n , such that $\mathcal{F}_{\underline{a}}(\bullet_d) = a_d$ for any $d \in D$.

The vector fields $\mathcal{F}_a(t)$ (or $\mathcal{F}_{\underline{a}}(t)$ in the coloured case) are the *elementary differentials*, building blocks of the B-series [16] which are defined as follows: for any linear form α on $\mathcal{T}_D \oplus \mathbb{R}\mathbf{1}$ where $\mathbf{1}$ is the empty tree, for any collection of vector fields \underline{a} and for any initial point $y_0 \in \mathbb{R}^n$, the corresponding B-series¹ is a formal series in the indeterminate h given by:

$$(3) \quad B_{\underline{a}}(\alpha, y_0) = \alpha(\mathbf{1})y_0 + \sum_{t \in \mathcal{T}_D} h^{|t|} \frac{\alpha(t)}{\text{sym}(t)} \mathcal{F}_{\underline{a}}(t)(y_0).$$

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¹Such coloured B-series are sometimes called NB-series in the literature.

Here $|t|$ is the number of vertices of t , and $\text{sym}(t)$ is its symmetry factor, i.e. the cardinal of its automorphism group $\text{Aut } t$. For any vector field a , the exact solution of the differential equation:

$$(4) \quad \dot{y}(t) = a(y(t))$$

with initial condition $y(0) = y_0$ admits a (one-coloured) B-series expansion at time $t = h$, and its approximation by any Runge-Kutta method as well [5, 6, 16]. The formal transformation $y_0 \mapsto B_a(\alpha, y_0)$ is a formal series with coefficients in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

We will be interested in *canonical B-series* [7], i.e. such that the formal transformation $B_{\underline{a}}(\alpha, -)$ is a symplectomorphism for any collection of hamiltonian vector fields \underline{a} . Here, the dimension $n = 2r$ is even, and \mathbb{R}^{2r} is endowed with the standard symplectic structure:

$$(5) \quad \omega(x, y) = \sum_{i=1}^r x_i y_{r+i} - x_{r+i} y_i,$$

and a vector field $a = \sum_{i=1}^{2r} a_i \partial_i$ is hamiltonian if there exists a smooth map $H : \mathbb{R}^{2r} \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} a_i &= -\frac{\partial H}{\partial t_{i+r}} \text{ for } i = 1, \dots, r, \text{ and} \\ a_i &= \frac{\partial H}{\partial t_{i-r}} \text{ for } i = r+1, \dots, 2r. \end{aligned}$$

Recall that the Poisson bracket of two smooth maps f, g on \mathbb{R}^{2r} is given by:

$$(6) \quad \{f, g\} = \sum_{i=1}^r \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_{i+r}} - \frac{\partial g}{\partial t_i} \frac{\partial f}{\partial t_{i+r}}.$$

Hence hamiltonian vector fields are those vector fields a which can be expressed as:

$$a = \{H, -\}$$

for some $H \in C^\infty(\mathbb{R}^{2r})$. A B-series turns out to be canonical if and only if the following condition holds for any rooted trees s and t [3, Theorem 2]:

$$(7) \quad \alpha(s \circ t) + \alpha(t \circ s) = \alpha(s)\alpha(t),$$

where $s \circ t$ is the right Butcher product, defined by grafting the tree t on the root of the tree s . This result is also valid in the coloured case. The infinitesimal counterpart of this result expresses as follows ([16], Theorem IX.9.10 for one-colour case): a B-series $B_{\underline{a}}(\alpha, -)$ with $\alpha(\mathbf{1}) = 0$ defines a hamiltonian vector field for any hamiltonian vector field a if and only if:

$$(8) \quad \alpha(s \circ t) + \alpha(t \circ s) = 0.$$

Let us call the B-series of the type described above *hamiltonian B-series*. Our interest in non-rooted trees comes from the following elementary observation: the two rooted trees $s \circ t$ and $t \circ s$ are equal as non-rooted trees, and one is obtained from the other by shifting the root to a neighbouring vertex. As an easy consequence of (8), any hamiltonian B-series $B_{\underline{a}}(\alpha, -)$ has to satisfy that if two rooted trees s and t are equal as non-rooted trees, then:

$$(9) \quad \alpha(s) = \pm \alpha(t).$$

This implies that, modulo a careful account of the signs involved, hamiltonian B-series are naturally indexed by non-rooted trees rather than by rooted ones. The sign is plus or minus according to the parity of the minimal number of "root shifts" $s_1 \circ s_2 \mapsto s_2 \circ s_1$ that are required to change s into t .

In the present paper we address the following question: *what survives from the pre-Lie structure at the level of non-rooted trees?* There is a natural linear map \tilde{X} from non-rooted trees to (the linear span of) rooted trees, sending a tree to the sum of all its rooted representatives, with alternating signs. Its precise definition involves a total order on rooted trees introduced by A. Murua [19]. We propose a binary product \diamond on the linear span of non-rooted trees, which is roughly speaking an alternating sum of all trees obtained by linking a vertex of the first tree with a vertex of the second tree. Theorem 4 is the key result of the paper. It implies the fact that \diamond is a Lie bracket and that \tilde{X} is a Lie algebra morphism, the Lie bracket on rooted trees being given by antisymmetrizing the pre-Lie product.

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2. STRUCTURAL FACTS ABOUT NON-ROOTED TREES

We denote by T (resp. FT) the set of non-planar rooted (resp. non-rooted) trees. We denote by \mathcal{T} (resp. \mathcal{FT}) the vector spaces freely generated by T (resp. FT). The projection $\pi : T \twoheadrightarrow FT$ is defined by forgetting the root. It extends linearly to $\pi : \mathcal{T} \twoheadrightarrow \mathcal{FT}$. Rooted trees will be denoted by latin letters s, t, \dots , non-rooted trees by greek letters σ, τ, \dots . We will also use "free tree" as a synonymous for "non-rooted tree". For any free tree τ and for any vertex v of τ , we denote by τ_v the unique rooted tree built from τ by putting the root at v .

2.1. A total order on rooted trees. Recall that any rooted tree t is obtained by grafting rooted trees t_1, \dots, t_q on a common root:

$$t = B_+(t_1, \dots, t_q).$$

The trees t_j are called the *branches* of t . A. Murua defines in [19] a total order on the set of (one-colour) rooted trees in a recursive way as follows: the *canonical decomposition* of a tree t is given by $t = t_L \circ t_R$ where t_R is the maximal branch of t . The maximality is to be understood with respect to the total order, supposed to be already defined for trees with number of vertices strictly smaller than $|t|$. Then $s < t$ if and only if:

- either $|s| < |t|$,
- or $|s| = |t|$ and $s_L < t_L$,
- or $|s| = |t|$, $s_L = t_L$ and $s_R < t_R$.

In the one-colour case, the total order of the first few trees is:

$$\bullet < \begin{array}{c} \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array} < \dots$$

If we prescribe a total order on the set of colours D and allow the set of one node coloured trees to inherit this order, incorporating this into the definition above gives a total order on the set of coloured rooted trees. Note that the structure of the one-colour order is not entirely preserved,

as, for example, for two colours $\bullet < \circ$ we have $\begin{array}{c} \bullet \\ | \\ \circ \end{array} > \begin{array}{c} \circ \\ / \backslash \\ \bullet \bullet \end{array}$ whereas $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} < \begin{array}{c} \bullet \\ / \backslash \\ \bullet \bullet \end{array}$.

2.2. Superfluous trees. This notion has been introduced in [1], where the authors describe order conditions for canonical B-series coming from Runge-Kutta approximation methods. Let $B_{\underline{a}}(\alpha, -)$ be a hamiltonian B-series. According to (8), we have $\alpha(t \circ t) = 0$ for any rooted tree t . Any non-rooted tree τ such that there exists a rooted tree s with $s \circ s \in \pi^{-1}(\tau)$ is called a *superfluous tree*, and a rooted tree t is said to be superfluous if its underlying free tree $\pi(t)$ is. Such trees never appear in a hamiltonian B-series. For any free tree $\tau \in FT$, its *canonical representative* is the maximal element of the set $\pi^{-1}(\tau) \subset T$ for the total order above. The following lemma gives a characterization of superfluous trees:

Lemma 1. *Let $\tau \in FT$ have two distinct vertices v and w such that $\tau_v = \tau_w$ is the canonical representative of τ . Then:*

- (1) *v and w are the two ends of a common edge in τ ,*
- (2) *There exists $s \in T$ such that $\tau_v = \tau_w = s \circ s$.*

Proof. First of all, the maximal branch of τ_v contains w (and vice-versa). Indeed, Suppose the maximal branch of τ_v does not contain w (and hence vice-versa). Let

$$\tau_v = B^+(t_1, t_2, \dots, t_n, t_w, t_{\max}), \quad \tau_w = B^+(t'_1, t'_2, \dots, t'_n, t'_v, t'_{\max}),$$

where t_w is the branch of τ_v containing w and t'_v similarly. It is clear that t'_v contains all branches of τ_v except t_w . Hence $|t'_v| > |t_1| + \dots + |t_n| + |t_{\max}|$ and as $|t_{\max}| = |t'_{\max}|$ we have $|t'_v| > |t'_{\max}|$, a contradiction. Now suppose that v and w are not neighbours, and choose a vertex x between v and w , i.e. such that there is a path from v to w of meeting x . The maximal branch of τ_x cannot contain both v and w ; suppose it does not contain v . Then it is a subtree of the maximal branch τ_v and hence contains strictly less vertices. Looking at the canonical decompositions:

$$t := \tau_v = \tau_w = t_L \circ t_R, \quad t' := \tau_x = t'_L \circ t'_R,$$

we have then $|t'_L| > |t_L|$, which immediately yields $\tau_x > \tau_v$, which is a contradiction. This proves the first assertion, and the second assertion follows immediately. \square

There are four superfluous free trees with six vertices or less. The corresponding superfluous rooted trees are:



We denote by S the set of superfluous free trees and by FT' the set of non-superfluous trees, hence $FT = FT' \amalg S$. The corresponding linear spans will be denoted by \mathcal{S} and \mathcal{FT}' . We have $\mathcal{FT} = \mathcal{S} \oplus \mathcal{FT}'$, which leads to a linear isomorphism:

$$\mathcal{FT}' \sim \mathcal{FT}/\mathcal{S}.$$

2.3. Symmetries. We keep the notations of the previous subsection. For any non-superfluous tree $\tau \in FT'$ we denote by $*$ the unique vertex such that τ_* is the canonical representative of τ . The group of automorphisms of τ is the subgroup $\text{Aut } \tau$ of the group of permutations φ of $\mathcal{V}(\tau)$ which respect the tree structure, i.e. such that, for any $v, w \in \mathcal{V}(\tau)$, there is an edge between v and w if and only if there is an edge between $\varphi(v)$ and $\varphi(w)$.

For any rooted tree t we also denote by $\text{Aut } t$ its group of automorphisms, i.e. the subgroup of the group of permutations φ of $\mathcal{V}(t)$ which respect the rooted tree structure. It obviously coincides with the stabilizer of the root in $\text{Aut } \pi(t)$. Now for any non-superfluous free tree τ it is obvious from Lemma 1 that $\text{Aut } \tau$ fixes the vertex $*$, hence $\text{Aut } \tau = \text{Aut } \tau_*$.

Now $\text{Aut } \tau$ acts on the set of vertices $\mathcal{V}(\tau)$. Moreover, for any vertex v this group acts transitively on the subset of possible roots for τ_v , namely:

$$\mathcal{R}_v(\tau) := \{w \in \mathcal{V}(\tau), \tau_w \sim \tau_v\}.$$

Hence $R_v(\tau)$ identifies itself with the homogeneous space:

$$(10) \quad R_v(\tau) \sim \text{Aut } \tau_* / \text{Aut } \tau_v.$$

This immediately leads to the following proposition, which is implicit in the proof of Lemma IX.9.7 in [16]:

Proposition 2. *Let τ be a non-superfluous free tree, let t be a rooted tree such that $\pi(t) = \tau$, and let $N(t, \tau)$ be the number of vertices $v \in \mathcal{V}(\tau)$ such that $\tau_v = t$. Then:*

$$(11) \quad N(t, \tau) = \frac{\text{sym}(\tau_*)}{\text{sym}(t)}.$$

2.4. Grafting and linking. Let σ and τ be two non-rooted trees, and let us choose a vertex v of σ and a vertex w of τ . We will denote by $\sigma_v \text{---}_w \tau$ the non-rooted tree obtained by taking σ and τ together and adding a new edge between v and w . This linking operation is related to grafting of rooted trees as follows: for any other choice of vertices x of σ and y of τ we have:

$$(12) \quad (\sigma_v \text{---}_w \tau)_y = \sigma_v \rightarrow_w \tau_y,$$

$$(13) \quad (\sigma_v \text{---}_w \tau)_x = \tau_w \rightarrow_v \sigma_x.$$

3. A BINARY OPERATION ON NON-ROOTED TREES

The linear map $\tilde{X} : \mathcal{FT} \rightarrow \mathcal{T}$ is defined for any non-rooted tree τ by:

$$(14) \quad \tilde{X}(\tau) = \sum_{v \in \mathcal{V}(\tau)} \varepsilon(v, \tau) \tau_v,$$

and extended linearly. Here $\varepsilon(v, \tau)$ is equal to 0 if τ is superfluous, and is equal to 1 (resp. -1) if τ is not superfluous and if the number of requested root shifts to change τ_v into the canonical representative of τ is even (resp. odd). This number, which we denote by $\kappa(v, \tau)$, is indeed unambiguous for non-superfluous trees according to Lemma 1. We obviously have:

$$(15) \quad \varepsilon(v, \tau) = \varepsilon(\varphi(v), \tau)$$

for any $\varphi \in \text{Aut } \tau$. The introduction of the map \tilde{X} is justified by the fact that, according to (8), (15) and Proposition 2, rooted trees involved in hamiltonian B-series do group themselves under terms $\tilde{X}(\tau)$ with $\tau \in FT$. Indeed,

Proposition 3.

$$(16) \quad B_{\underline{a}}(\alpha, -) = \sum_{\tau \in FT} h^{|\tau|} \frac{\alpha(\tau_*)}{\text{sym}(\tau_*)} \mathcal{F}_{\underline{a}}(\tilde{X}(\tau)).$$

Now let us define a binary product on \mathcal{FT} by the formula:

$$(17) \quad \sigma \diamond \tau = \sum_{v \in \mathcal{V}(\sigma), w \in \mathcal{V}(\tau)} \delta(v, w) \sigma_v \text{---}_w \tau,$$

with $\delta(v, w) := \varepsilon(w, \sigma_v \text{---}_w \tau) \varepsilon(v, \sigma) \varepsilon(w, \tau)$.

Theorem 4. *We have $\sigma \diamond \tau \in \mathcal{FT}'$ for any $\sigma, \tau \in \mathcal{FT}$, and $\sigma \diamond \tau = 0$ if σ or τ is superfluous. The product \diamond is antisymmetric, and the following relation holds:*

$$(18) \quad \tilde{X}(\sigma \diamond \tau) = \tilde{X}(\sigma) \rightarrow \tilde{X}(\tau) - \tilde{X}(\tau) \rightarrow \tilde{X}(\sigma) = [\tilde{X}(\sigma), \tilde{X}(\tau)].$$

Proof. A computation of the left-hand side gives:

$$\begin{aligned}\tilde{X}(\sigma \diamond \tau) &= \sum_{v, x \in \mathcal{V}(\sigma), w \in \mathcal{V}(\tau)} \varepsilon(x, \sigma_{v \text{---} w} \tau) \varepsilon(w, \sigma_{v \text{---} w} \tau) \varepsilon(v, \sigma) \varepsilon(w, \tau) (\sigma_{v \text{---} w} \tau)_x \\ &+ \sum_{v \in \mathcal{V}(\sigma), w, y \in \mathcal{V}(\tau)} \varepsilon(y, \sigma_{v \text{---} w} \tau) \varepsilon(w, \sigma_{v \text{---} w} \tau) \varepsilon(v, \sigma) \varepsilon(w, \tau) (\sigma_{v \text{---} w} \tau)_y,\end{aligned}$$

and computing the right-hand side gives:

$$\begin{aligned}[\tilde{X}(\sigma), \tilde{X}(\tau)] &= - \sum_{v, x \in \mathcal{V}(\sigma), w \in \mathcal{V}(\tau)} \varepsilon(v, \sigma) \varepsilon(w, \tau) \tau_w \rightarrow_x \sigma_v \\ &+ \sum_{v \in \mathcal{V}(\sigma), w, y \in \mathcal{V}(\tau)} \varepsilon(v, \sigma) \varepsilon(w, \tau) \sigma_v \rightarrow_y \tau_w.\end{aligned}$$

Exchanging x and v in the first sum, and y and w in the second, we get:

$$\begin{aligned}[\tilde{X}(\sigma), \tilde{X}(\tau)] &= - \sum_{v, x \in \mathcal{V}(\sigma), w \in \mathcal{V}(\tau)} \varepsilon(x, \sigma) \varepsilon(w, \tau) \tau_w \rightarrow_v \sigma_x \\ &+ \sum_{v \in \mathcal{V}(\sigma), w, y \in \mathcal{V}(\tau)} \varepsilon(v, \sigma) \varepsilon(y, \tau) \sigma_v \rightarrow_w \tau_y.\end{aligned}$$

The first assertion is immediate since $\varepsilon(w, \sigma_{v \text{---} w} \tau)$ vanishes if $\sigma_{v \text{---} w} \tau$ is superfluous. The second assertion is also immediate, since $\delta(v, w)$ vanishes if σ or τ is superfluous. The antisymmetry comes from the fact that v and w are neighbours in $\sigma_{v \text{---} w} \tau$.

- (1) If σ or τ is superfluous, any individual term in both sides vanishes.
- (2) If σ and τ are not superfluous it may happen that $\sigma_{v \text{---} w} \tau$ is superfluous for some $v \in \mathcal{V}(\sigma)$ and $w \in \mathcal{V}(\tau)$. The corresponding term $\tilde{X}(\sigma_{v \text{---} w} \tau)$ in $\tilde{X}(\sigma \diamond \tau)$ vanishes. On the other hand, the sum of all terms in $[\tilde{X}(\sigma), \tilde{X}(\tau)]$ corresponding to the couple (v, w) chosen above writes down as:

$$\begin{aligned}T_{v, w} &:= - \sum_{x \in \mathcal{V}(\sigma)} (-1)^{\kappa(x, \sigma) + \kappa(w, \tau)} \tau_w \rightarrow_v \sigma_x \\ &+ \sum_{y \in \mathcal{V}(\tau)} (-1)^{\kappa(v, \sigma) + \kappa(y, \tau)} \sigma_v \rightarrow_w \tau_y.\end{aligned}$$

The distance $d(x, v)$ between x and v in σ is defined as the length of the (unique) path joining x and v in σ . It is clearly equal modulo 2 to the sum $\kappa(x, \sigma) + \kappa(v, \sigma)$. Similarly, $d(y, w) = \kappa(y, \tau) + \kappa(w, \tau)$ modulo 2. Hence, using (12) and (13) we get:

$$T_{v, w} = (-1)^{\kappa(v, \sigma) + \kappa(w, \tau)} \left(- \sum_{x \in \mathcal{V}(\sigma)} (-1)^{d(x, v)} (\sigma_{v \text{---} w} \tau)_x + \sum_{y \in \mathcal{V}(\tau)} (-1)^{d(y, w)} (\sigma_{v \text{---} w} \tau)_y \right).$$

Now the distance $d(x, v)$ is the same if we compute it in σ or in $\sigma_{v \text{---} w} \tau$, and similarly for $d(y, w)$. Finally, using the fact that v and w are neighbours in $\sigma_{v \text{---} w} \tau$, we have $d(x, w) = d(x, v) + 1$ for any $x \in \mathcal{V}(\sigma)$, the distance being computed in $\sigma_{v \text{---} w} \tau$. This finally gives:

$$T_{v, w} = (-1)^{\kappa(v, \sigma) + \kappa(w, \tau)} \sum_{z \in \mathcal{V}(\sigma_{v \text{---} w} \tau)} (-1)^{d(z, w)} (\sigma_{v \text{---} w} \tau)_z,$$

which vanishes since $\sigma_{v \text{---} w} \tau$ is superfluous.

- (3) Finally, if σ , τ and $\sigma_{v\text{---}w}\tau$ are not superfluous, using (12) and (13), both sides will be equal if we have:

$$\begin{aligned}\kappa(x, \sigma_{v\text{---}w}\tau) + \kappa(w, \sigma_{v\text{---}w}\tau) + \kappa(v, \sigma) &= \kappa(x, \sigma) + 1 \text{ modulo } 2, \\ \kappa(y, \sigma_{v\text{---}w}\tau) + \kappa(w, \sigma_{v\text{---}w}\tau) + \kappa(w, \tau) &= \kappa(y, \tau) \text{ modulo } 2.\end{aligned}$$

Using the fact that v and w are neighbours, it rewrites as:

$$\begin{aligned}\kappa(x, \sigma_{v\text{---}w}\tau) + \kappa(x, \sigma) &= \kappa(v, \sigma_{v\text{---}w}\tau) + \kappa(v, \sigma) \text{ modulo } 2, \\ \kappa(y, \sigma_{v\text{---}w}\tau) + \kappa(y, \tau) &= \kappa(w, \sigma_{v\text{---}w}\tau) + \kappa(w, \tau) \text{ modulo } 2.\end{aligned}$$

These two last identities are always verified: looking for example at the right-hand side of the first one, moving vertex v to a neighbour will change both κ 's by ± 1 . It remains then to jump from neighbour to neighbour up to x . The proof of the second identity is completely similar. □

Using the identification of \mathcal{FT}/\mathcal{S} with \mathcal{FT}' , a straightforward consequence of Theorem 4 is the following:

Corollary 5. *The linear map \tilde{X} is an injection of \mathcal{FT}' into \mathcal{T} , and the product $\diamond : \mathcal{FT}' \times \mathcal{FT}' \rightarrow \mathcal{FT}'$ verifies:*

$$\tilde{X}(\sigma \diamond \tau) = [\tilde{X}(\sigma), \tilde{X}(\tau)].$$

As a consequence, the product \diamond satisfies the Jacobi identity, and \tilde{X} is an embedding of Lie algebras.

4. APPLICATION TO ELEMENTARY HAMILTONIANS

Keeping the previous notations, the vector field $\mathcal{F}_{\underline{a}}(\tilde{X}(\tau))$ is hamiltonian for any (decorated) non-rooted tree τ . Hence it can be uniquely written as $\{H_{\underline{a}}(\tau), -\}$ for some $H_{\underline{a}}(\tau) \in C^\infty(\mathbb{R}^{2r})$, called the *elementary hamiltonian* associated with τ .

Proposition 6. *For any free trees σ, τ we have:*

$$(19) \quad \{H_{\underline{a}}(\sigma), H_{\underline{a}}(\tau)\} = H_{\underline{a}}(\sigma \diamond \tau).$$

Proof. We compute:

$$\begin{aligned}\{\{H_{\underline{a}}(\sigma), H_{\underline{a}}(\tau)\}, -\} &= [\{H_{\underline{a}}(\sigma), -\}, \{H_{\underline{a}}(\tau), -\}] \\ &= [\mathcal{F}_{\underline{a}}(\tilde{X}(\sigma)), \mathcal{F}_{\underline{a}}(\tilde{X}(\tau))] \\ &= \mathcal{F}_{\underline{a}}([\tilde{X}(\sigma), \tilde{X}(\tau)]) \\ &= \mathcal{F}_{\underline{a}} \circ \tilde{X}(\sigma \diamond \tau) \\ &= \{H_{\underline{a}}(\sigma \diamond \tau), -\}.\end{aligned}$$

One concludes by using the uniqueness of the hamiltonian representation of a hamiltonian vector field. □

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